SOME METHODS FOR SOLVING A SINGLE NONLINEAR INTEGRO-DIFFERENTIAL EQUATION OF THE THEORY OF JETS OF AN IDEAL FLUID

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The following integro-differential equation is derived in [1]:

$$u'(\xi) = \lambda\gamma(\xi)U(u(\xi) + \alpha(\xi))V(-T(u\mid\xi) + \beta(\xi)) + \delta(\xi), \quad \xi \text{ on } [-1,1] \qquad (A)$$

in which $\alpha(\xi)$, $\beta(\xi)$, $\gamma(\xi)$, $\delta(\xi)$, $U(u+\alpha)$, $V(-T+\beta)$ are given functions of their own arguments, λ is a given parameter, and $T(u\mid\xi)$ is a singular integral of the form

$$T(u\mid\xi) = \frac{\omega(\xi)}{\pi}\int_{-1}^{1}\frac{u(t)}{t-\xi}\frac{dt}{\omega(t)}, \qquad \xi \text{ on } [-1, 1]. \qquad (0.1)$$

Here $\omega(\xi)$ is a given function. The notation $T(u\mid\xi)$ serves to emphasize that the given integral is not only a function of the point ξ, but is also an operator in $u(\xi)$. A large class of problems for plane steady jet flows of an ideal incompressible fluid reduce to equation (A). In this article, we discuss several methods for solving equation (A). The proposed methods are applied to jet flows with a curvilinear wall and to jet flows of a heavy fluid with rectilinear boundaries. The notation $T(u)$ and $T(u, \omega\mid\xi)$ are utilized for the operator $T(u\mid\xi)$ when it is necessary to emphasize its independence of $u(\xi)$ or to note dependence on $\omega(\xi)$. Analogous notation is also used for other operators encountered in the article. In cases when $\omega(\xi) \equiv 1$ and $\omega(\xi) = \sqrt{1-\xi^2}$, $T(u\mid\xi)$ is denoted by

$$J(u\mid\xi) = \frac{1}{\pi}\int_{-1}^{1}\frac{u(t)}{t-\xi}dt,$$

$$I(u\mid\xi) = \frac{\sqrt{1-\xi^2}}{\pi}\int_{-1}^{1}\frac{u(t)}{t-\xi}\frac{dt}{\sqrt{1-t^2}} \quad \xi \text{ on } [-1, 1]. \qquad (0.2)$$

A part of the results of this article were reported at the Second Congress on Theoretical and Applied Mechanics [2].

§1. Small parameter methods. The linearization method.

1.1. Reduction of Eq. (A) to a functional equation in Banach space. We introduce the operator (ξ, ξ_0 on $[-1, 1]$)

$$S(u) = S(u\mid\xi) \equiv \lambda\int_{\xi_0}^{\xi}\gamma(t)U(u(t) + \alpha(t)) \times$$
$$\times V(-T(u\mid t) + \beta(t))\,dt + \int_{\xi_0}^{\xi}\delta(t)\,dt. \qquad (1.1)$$

If the desired solution of Eq. (A) satisfies the condition

$$u(\xi_0) = 0, \qquad \xi_0 \text{ on } [-1, 1], \qquad (1.2)$$

then Eq. (A) can be written in the form

$$u = S(u) = S(u\mid\xi). \qquad (1.3)$$

Henceforth, it will always be assumed that condition (1.2) is satisfied. When the operator $S(u)$ is defined on some Banach space, equation (1.3) will be a functional equation in this space. In functional analysis, various methods of successive approximations have been developed for solving such equations; thus it is natural to apply these methods to equation (A) written in form (1.3). Since the application of these methods depends on the space in which the operator $S(u)$ is written, it is necessary first of all to consider the problem of the class of the desired functions.

1.2. The class of desired functions. Some bounds. Let $\rho(\xi)$ be a positive continuous function given on the segment $[-1, 1]$ such that the function $1/\rho(\xi)$ is integrable on the segment $[-1, 1]$. We shall use C_ρ to denote the class of functions defined on segment $[-1, 1]$ and which satisfy the condition that the product of any function of this class $u(\xi)$ and $\rho(\xi)$ is continuous on the given segment. It is obvious that if we introduce the norm

$$|u|_\rho = \max|\rho(\xi)u(\xi)| \qquad (1 \leqslant \xi \leqslant 1),$$

the class C_ρ will be a Banach space. We shall use C_ρ^1 to denote the class of functions defined on the segment $[-1, 1]$ and satisfying the conditions: (1) any function of this class $u(\xi)$ satisfies condition (1.2); (2) the product of $u'(\xi)$ and $\rho(\xi)$ is continuous on the segment $[-1, 1]$. It is not difficult to show that the class C_ρ^1 will be a Banach space if we introduce the norm

$$\|u\|_\rho = \max|\rho(\xi)u'(\xi)| \qquad (1 \leqslant \xi \leqslant 1).$$

When $\rho(\xi) \equiv 1$, then the class C_ρ coincides with the known class C of continuous functions, and the class C_ρ^1 will be a closed set of the known class C^1 of continuously differentiable functions; in this special case, the following notation will be used for the norms $|u|_\rho$ and $\|u\|_\rho$:

$$|u|_1 = \max|u(\xi)|, \qquad \|u\|_1 = \max|u'(\xi)|, \qquad (1 \leqslant \xi \leqslant 1).$$

The introduction of the spaces C_ρ and C_ρ^1 makes it possible to obtain some bounds which permit justification of the methods presented below. We shall impose the following condition on $\rho(\xi)$ and the function $\omega(\xi)$ included in $T(u|\xi)$:

$$T(u\mid\xi) = \int_{-1}^{1}H(\omega\mid\xi, t)\,u'(t)\,dt.$$

Here $H(\omega|\xi, t)$ is a Fredholm-type kernel such that the function $H(\omega|\xi, t)/(t)$ is absolutely integrable with respect to t; however, the integral of the modulus of this function is continuous for ξ. Then, we introduce the quantities

$$a = a(\rho) = \max\left|\int_{\xi_0}^{\xi}\frac{dt}{\rho(t)}\right|,$$

$$b = b(\rho, \omega) = \max\int_{-1}^{1}\left|\frac{H(\omega\mid\xi, t)}{\rho(t)}\right|dt \quad (-1 \leqslant \xi_0 \leqslant 1).$$

Let $u(\xi) \in C_\rho^1$ and the conditions imposed previously on $\rho(\xi)$ and $\omega(\xi)$ be satisfied; then it is not difficult to show that the following inequalities hold:

$$|u|_1 \leqslant a(\rho)\|u\|_\rho, \qquad |T(u)|_1 \leqslant b(\rho,\omega)\|u\|_\rho. \quad (1.5)$$

If we consider the integrals of the products $p(\xi)u(\xi)$ and $p(\xi)T(u|\xi)$, where $p(\xi) \in C_\rho$, then, making use of inequality (1.5), it is easy to obtain the bounds

$$\left\|\int_{\xi_0}^{\xi} pu\,dt\right\|_\rho \leqslant a(\rho)|p|_\rho\|u\|_\rho,$$

$$\left\|\int_{\xi_0}^{\xi} pT(u)\,dt\right\|_\rho \leqslant b(\rho,\omega)|p|_\rho\|u\|_\rho. \quad (1.6)$$

It is obvious that the quantity $a(\rho)$ for a given class C_ρ can always be computed with a given accuracy. We cite the values of of $a(\rho)$ for different classes C_ρ^1

$$\rho(\xi) \equiv 1, \quad \rho(\xi) = \sqrt{1-\xi^2}, \quad \rho(\xi) = \sqrt{1 \pm \xi},$$
$$a(\rho) = 2, \qquad \pi, \qquad 2\sqrt{2}.$$

For these same classes, we cite expressions for quantities $b(\rho, \omega)$:

$\omega(\xi) = \sqrt{1-\xi^2}$,	$\omega(\xi) \equiv 1$	
$b(\rho,\omega) = 1$,	$\frac{2}{\pi}(1+2\ln 2)$	when $\rho(\xi) = 1$
$b(\rho,\omega) = 4G/\pi$,	$3\ln 2$	when $\rho(\xi) = \sqrt{1-\xi^2}$
$b(\rho,\omega) = \frac{4}{\pi}\sqrt{2(H^2-1)}$,	$\frac{2\sqrt{2}}{\pi}(2+\ln 2)$	when $\rho(\xi) = \sqrt{1 \pm \xi}$

When computing $b(\rho, \omega)$ in the case $\omega(\xi) \equiv 1$, it was assumed that $u(\xi)$ satisfies the additional condition $u(1) = u(-1) = 0$.

Here G is the Catalan constant, $G = 0.915965594\ldots$, H is the root of the equation $t - \mathrm{Arth}(1/t) = 0$, $H = 1.199678402\ldots$ It should be noted that the quantity $b(\rho, \omega)$ was computed to four decimal places by Ya. I. Sekerzh-Zen'kovich in the case $\rho(\xi) = \omega(\xi) = \sqrt{1-\xi^2}$ [3]. It should also be noted that if the quantity $a(\rho)$ is introduced as the maximum of the integral included in (1.4) for a given value of ξ_0, this quantity can be decreased. As for the coefficients of equation (A), it will be assumed henceforth that $\gamma(\xi)$ and $\delta(\xi)$ belong to the class C_ρ, and $\alpha(\xi)$ and $\beta(\xi)$ are continuous functions. The solution of equation (A) will be sought in class C_ρ^1 in which the function $\rho(\xi)$ is the same as in class C_ρ.

1.3. Basic parameters. Majorants. We shall introduce some constant parameters characterizing Eq. (A). First of all, we have the parameter λ; it will be assumed always that $\lambda > 0$ (the latter is no restriction whatever on λ, since λ can always be made a positive quantity through selection of the function $\gamma(\lambda)$). In order to introduce the remaining parameters, we shall give the concept of the majorant of Eq. (A).

Let the functions $U(u + \alpha(\beta))$ and $V(u + \beta(\beta))$ be defined on the segment $-t' \le u \le t'$ for any ξ on $[-1, 1]$ and twice differentiable on this segment. Further, let the positive monotonically nondecreasing differentiable functions $U_\nu(t)$, $V_\nu(t)$ ($\nu = 0, 1, 2$), given on the segment $0 \le t < \infty$, majorize the functions $U(u + \alpha(\xi))$ and $V(u + \beta(\xi))$ and their derivatives on the segment $0 \le t \le t'$ in the following manner:

$$|U^\nu(u+\alpha(\xi))| \leqslant U_\nu(|u|),$$
$$|V^\nu(-u+\beta(\xi))| \leqslant V_\nu(|u|),$$
$$(\nu = 0,1,2;\ 0 \leqslant |u| \leqslant t'). \quad (1.7)$$

Then we have for the variable τ on $[0, \infty)$

$$M(\tau) = \frac{|\gamma|_\rho}{\eta}[aU_1(a\sigma\tau)V_0(b\sigma\tau) + bU_0(a\sigma\tau)V_1(b\sigma\tau)], \quad (1.8)$$

$$N(\tau) = \sigma'\frac{|\gamma|_\rho}{\eta'}[a^2U_2(a\sigma'\tau)V_0(b\sigma'\tau) + 2abU_1(a\sigma'\tau)V_1(b\sigma'\tau) + b^2U_0(a\sigma'\tau)V_2(b\sigma'\tau)], \quad (1.9)$$

where a, b are constants of (1.4) and a η, η', σ, σ' are parameters chosen so that the following condition is satisfied:

$$M(0) = M'(0) = N(0) = N'(0) = 1 \quad (1.10)$$

will be called henceforth majorants of Eq. (A). In this case, it will be assumed by definition that the derivative $M'(\tau)$ will be a positive and monotonically nondecreasing function.

The relationship (1.10) includes four equations which are readily solved for the parameters η, η', σ, σ'. We shall make use of them to form two more parameters $\varkappa$ and $\varkappa'$ defined by the equalities

$$\varkappa^2 = \frac{|\lambda U(\alpha)V(\beta)+\delta|_\rho}{\lambda\sigma\eta}, \qquad \varkappa'^2 = \frac{|\lambda U(\alpha)V(\beta)+\delta|_\rho}{\lambda\sigma'\eta'}. \quad (1.11)$$

It is easy to see that only five of the introduced parameters are independent and these can be taken as the basic parameters. Henceforth, we shall take λ, η, $\varkappa$, η', $\varkappa'$ as the basic parameters. We note that the parameters $\varkappa$ and $\varkappa'$ will not depend on λ if $\delta(\xi) \equiv 0$ or $\delta(\xi) = \lambda\delta^*(\xi)$.

1.4. The derivatives of S(u). Let us consider the problem of the existence of the Frechet derivatives of the operator S(u) and an estimate of their norms in C_ρ^1 space.

Theorem 1. Let the given functions included in Eq. (A) satisfy the following conditions: (1) the functions $\gamma(\xi)$ and $\delta(\xi)$ belong to class C_ρ; (2) functions $U(u+\alpha(\xi))$, $V(u+\beta(\xi))$ are defined and continuous on the segment

$$-t' \leqslant u \leqslant t', \qquad t' = mr, \qquad m = \max(a, b) \quad (1.12)$$

for any ξ on $[-1, 1]$. Then the operator S(u) will bring the sphere Ω_r: $0 \le \|u\|_\rho \le r$ of space C_ρ^1 into C_ρ^1. If in this case, the functions $U(u+\alpha(\xi))$, $V(u+\beta(\xi))$ are continuously differentiable in respect to u on the segment $[-1, 1]$ for any ξ on $[-1, 1]$, then the operator S(u) has at every point Ω_r a Frechet derivative defined by the equality

$$S'(u|\xi)u^* = \lambda\int_{\xi_0}^{\xi}\gamma(t)[U'(u(t) + \alpha(t))V(-T(u|t)+\beta(t))u^*(t) - U(u(t)+\alpha(t))V'(-T(u|t)+\beta(t))T(u^*|t)]\,dt. \quad (1.13)$$

In this case, the following bound holds for the norm $\|S'(u)\|_\rho$ of the linear operator $S'(u)u^*$:

$$\|S'(u)\|_\rho \leqslant \lambda\eta M(\|u\|_\rho/\sigma), \qquad u(\xi) \in \Omega_r, \quad (1.14)$$

where $M(\tau)$ is the majorant of Eq. (A) and σ is a parameter determined from Eqs. (1.10). If, however, the functions $U(u+\alpha(\xi))$, $V(u+\beta(\xi))$ are twice continuously differentiable with respect to u on the segment [−t', t'] for any ξ on [−1, 1], the operator S(u) has a second Frechet derivative in Ω_r:

$$S''(u\mid\xi)u^*u^{**}=\lambda\int_{\xi_0}^{\xi}\gamma(t)[U''(u(t)+\alpha(t))V(-T(u\mid t)+$$
$$+\beta(t))u^*(t)u^{**}(t)-U'(u(t)+\alpha(t))V'(-T(u\mid t)+$$
$$+\beta(t))(u^*(t)T(u^{**}\mid t)+u^{**}(t)T(u^*\mid t))+U(u(t)+$$
$$+\alpha(t))V''(-T(u\mid t)+\beta(t))T(u^*\mid t)T(u^{**}\mid t)]dt, \quad (1.15)$$

and the following bound holds for the norm $\|S''(u)\|_\rho$ of the bilinear operator $S''(u)u^*u^{**}$:

$$\|S''(u)\|_\rho\leqslant\lambda\eta'\frac{1}{\sigma'}N\left(\frac{\|u\|_\rho}{\sigma'}\right),\quad u(\xi)\in\Omega_r, \quad (1.16)$$

where N(τ) is the majorant of Eq. (A) and σ' is a parameter determined from Eqs. (1.10).

Proof. We shall establish, first of all, that the operator S(u) operates from Ω_r into C_ρ^1. This means that if we regard the operator $S(u)=S(u\mid\xi)$ as a function of ξ, then $S(u\mid\xi)\in C_\rho^1$, if $u(\xi)\in\Omega_r$. We consider the product

$$\rho(\xi)S_\xi'(u\mid\xi)=\lambda\rho(\xi)\{\gamma(\xi)U(u(\xi)+$$
$$+\alpha(\xi))V(-T(u\mid\xi)+\beta(\xi))+\delta(\xi)\}.$$

The functions $\rho(\xi)\gamma(\xi)$ and $\rho(\xi)\delta(\xi)$ included in this product are continuous, since γ(ξ) and δ(ξ) belong to class C_ρ by condition. The functions $U(u(\xi)+\alpha(\xi))$ and $V(-T(u\mid\xi)+\beta(\xi))$ are also continuous if $u(\xi)\in\Omega_r$, for, on the one hand, the functions $U(u+\alpha(\xi))$ and $V(u+\beta(\xi))$ are by condition continuous on the segment $\omega\leqslant|u|\leqslant mr$ for any ξ on [−1, 1] and, on the other hand, on the strength of inequalities (1.5), the functions u(ξ) and T(u | ξ) are continuous on the segment [−1, 1] and $0\leqslant|u|_1$, $|T(u)|_1\leqslant rm$, if $u(\xi)\in\Omega_r$. Thus, the product $\rho(\xi)\ S_\xi'(u\mid\xi)$ is a continuous function on the segment [−1, 1] and this means that $S(u\mid\xi)\in C_\rho^1$.

To verify the existence of the first and second Frechet derivatives of the operator S(u) for which formulas (1.13) and (1.15) are valid, it is sufficient to show that when the conditions imposed on the functions $U(u+\alpha(\xi))$ and $V(u+\beta(\xi))$ are satisfied, the following limiting equalities hold [6]:

$$\lim_{\|h\|_\rho\to0}\frac{\|\Delta(u;h)\|_\rho}{\|h\|_\rho}=0,\qquad\lim_{\|h^*\|_\rho\to0}\frac{\|\Delta(u;h,h^*)\|_\rho}{\|h^*\|_\rho}=0,\quad(1.17)$$

where $u(\xi)$, $h(\xi)$, $h^*(\xi)$ are arbitrary functions belonging to Ω_r, the operators

$$\Delta(u;h)=S(u+h)-S(u)-S'(u)h,$$
$$\Delta(u;h,h^*)=S'(u+h^*)h-S'(u)h-S''(u)hh^*.$$

We shall prove the first of the limiting equalities (1.17). Let us consider a function of two variables $f(x,y)=U(x+\alpha(\xi))V(y+\beta(\xi))$, continuously differentiable in the square $-t'\le x,\ y\le t'$. The following formula holds for this function:

$$f(x+\Delta x,\ y+\Delta y)-f(x,y)=f'(x+\theta\Delta x,\ y+\theta\Delta y)\ \Delta x+$$
$$+f_y'(x+\theta\Delta x,\ y+\theta\Delta y)\Delta y,$$
$$(0<\theta<1,\ -t'\leqslant x,\ x+\Delta x,\ y,\ y+\Delta y\leqslant t').$$

Let $u(\xi)\in\Omega_r$ and $h(\xi)\in\Omega_r$; at the same time, $u(\xi)+h(\xi)\in\Omega_r$. Then, on the strength of the inequalities (1.5), $0\leqslant|u|_1$, $|u+h|_1$, $|T(u)|_1$, $|T(u)+T(h)|_1\leqslant mr=t'$. Now, taking account of the last inequality, if we use the finite increments $x=u(\xi)$, $\Delta x=h(\xi)$, $y=T(u\mid\xi)$, $\Delta y=T(h\mid\xi)$ in the formula, then transform the operator Δ(u; h) with the aid of the expression obtained here, the operator can be represented in the form

$$\Delta(u,h)\equiv\lambda\int_{\xi_0}^{\xi}\gamma(t)\{[f_x'(u(t)+\theta h(t),\ -T(u\mid t)-\theta T(h\mid t))-$$
$$-f_x'(u(t),-T(u\mid t))h(t)-[f_y'(u(t)+\theta h(t),-T(u\mid t)-$$
$$-\theta T(h\mid t))-f_y'(u(t),\ -T(u\mid t))]T(h\mid t)\}\,dt\ .$$

The following inequality can be readily obtained with the aid of the bounds (1.6):

$$\|\Delta(u;\ h)\|_\rho/\|h\|_\rho\leqslant\lambda|\gamma|_\rho'[a|f_x'(u+\theta h,\ -T(u)-\theta T(h))-$$
$$-f_x'(u,-T(u))|_1+b|f_y'(u+\theta h,\ -\ T(u)-\theta T(h))-$$
$$-f_y'(u,-T(u))|_1].$$

Since the functions $f_x'(x,y)$ and $f_y'(x,y)$ are continuous in the square $-t'\le x,\ y\le t'$, the right-hand side of the obtained inequality tends to zero as $\|h\|_\rho\to0$. The second limiting equality of (1.17) is proved in a like manner.

We now differentiate equality (1.13) with respect to ξ, then multiply by ρ(ξ). We estimate the right-hand sides of the equality by the absolute value, introducing the majorizing functions $U_\nu(t)$, $V_\nu(t)$, (ν = 0, 1) and making use of inequalities (1.5). As a result, we have

$$\|S'(u\mid\xi)u^*\|_\rho\leqslant\lambda|\gamma|_\rho[aU_1(a\|u\|_\rho)V_0(b\|u\|_\rho)+$$
$$+bU_0(a\|u\|_\rho)V_1(b\|u\|_\rho)]\|u^*\|_\rho,\quad u(\xi)\in\Omega_r\ .$$

It is easy to see from the obtained inequality that the bound (1.14) holds for the norm of the operator $S'(u)u^*$. Proceeding in a like manner with the equality (1.15), we readily verify the validity of the bound (1.16) for the norm of the operator $S''(u)u^*u^{**}$.

1.5. The linearized equation. Let P(u) = 0 be a nonlinear equation given in Banach space. Then the linear equation $P'(u_0)(u_0-u)=P(u_0)$, where $P'(u_0)$ is a Frechet derivative at the point u_0 is said to be linearized at point u_0 or simply linearized. We write Eq. (1.3) in the form

$$P(u)=0,\qquad P(u)=P(u\mid\xi)\equiv u(\xi)-S(u\mid\xi).\ (1.18)$$

The equation linearized at point $u_0(\xi)\equiv0$ corresponding to equation (1.18) and, consequently, to Eq. (1.3) will be

$$P'(0\mid\xi)u\equiv u(\xi)-S'(0\mid\xi)u=F(\xi),$$
$$F(\xi)=S(0\mid\xi)\ ,\qquad(1.19)$$

where S'(0|ξ) is a Frechet derivative determined by formula (1.13). If Eq. (1.19) is solvable with any right side F(ξ) and its solution can be represented in the form

$$u(\xi)=\Gamma(F)=\Gamma(F\mid\xi),\qquad(1.20)$$

where Γ(F|ξ) is a linear operator, then the norm of this operator $\|\Gamma(F)\|_\rho$ will always be denoted by η_0 henceforth. It is known that the quantity η_0 should satisfy the inequality

$$\|\Gamma(F)\|_\rho\leqslant\eta_0\|F\|_\rho\ .\qquad(1.21)$$

1.6. Majorant equations. The majorants M(τ) and N(τ) determined by equalities (1.8) and (1.9) will be actual given functions if the majorizing functions U(t) and V(t) are constructed for Eq. (A).

We shall consider, along with Eq. (A), several transcendental equations which contain the majorants $M(\tau)$ and $N(\tau)$. Three of them contain the majorant $M(\tau)$,

$$\tau = \varkappa / \sqrt{M'(\tau)} \qquad (0 \leqslant \tau < \infty), \tag{1.22}$$

$$\tau M(\tau) - \int_0^\tau M(\tau)\,d\tau - \varkappa^2 = 0 \qquad (0 \leqslant \tau < \infty), \tag{1.23}$$

$$\tau = \varphi(\tau), \varphi(\tau) \equiv \nu\left(\int_0^\tau M(\tau)\,d\tau + \varkappa^2\right)(0 \leqslant \tau < \infty,\ \nu = \lambda\eta), \tag{1.24}$$

and the two others, the majorant $N(\tau)$,

$$\tau \int_0^\tau N(\tau)\,d\tau - \int_0^\tau\int_0^\tau N(\tau)\,d\tau\,d\tau - \varkappa'^2 = 0 \quad (0 \leqslant \tau < \infty) \tag{1.25}$$

$$\psi(\tau) = 0, \quad \psi(\tau) \equiv -\tau + \nu'\left(\int_0^\tau\int_0^\tau N(\tau)\,d\tau\,d\tau + \varkappa'^2\right)$$

$$(0 \leqslant \tau < \infty,\ \nu' = \lambda\eta'\eta_0) \tag{1.26}$$

We shall dwell on the problem of the existence of roots of these equations. Since $1/M'(\tau)$ is a positive monotonically non-increasing function, it is obvious that equation (1.22) has a single root. It is not difficult to show that equations (1.23) and (1.25) also have a single root. Indeed, if the left-hand side of equation (1.23) is denoted by $f(\tau)$, then it is easy to see that $f'(\tau) = \tau M'(\tau) > 0$, when $\tau > 0$; at the same time $f(0) = -\varkappa^2 < 0$, which implies that there exists one and only one point at which the function $f(\tau)$ vanishes. In precisely the same way, if the right-hand side of equation (1.25) is denoted by $f^*(\tau)$, it is easy to verify that $f_*'(\tau) = \tau N(\tau) > 0$ when $\tau > 0$; at the same time, $f_*(0) = -\varkappa'^2 < 0$, which implies the existence of a single zero of the function $f^*(\tau)$. Depending on the values of ν and ν', equations (1.24) and (1.26) can have two roots, or also have no roots at all. The following lemma holds for the least root of equation (1.24).

Lemma 1. If $\nu \leqslant \nu_0 = \tau_0\nu / \varphi(\tau_0) = 1 / M(\tau_0)$, where τ_0 is the root of equation (1.23) and $\varphi(\tau)$ is a function contained in the right-hand side of equation (1.24), equation (1.24) has on the segment $[0, \tau_0]$ the single root τ^* to which the following sequence converges:

$$\tau_0' = 0, \quad \tau_{n+1}' = \varphi(\tau_n') \qquad (\tau = 0, 1, \ldots), \tag{1.27}$$

the rate of convergence of this sequence being characterized by the inequality

$$\tau_* - \tau_n' \leqslant h_0^n\, \tau_*, \quad h_0 = \lambda / \eta M(\tau_0) \qquad (n = 0, 1, \ldots). \tag{1.28}$$

Proof. We write equation (1.24) in the form $\nu = f(\tau)$, $f(\tau) \equiv \equiv \tau\nu/\varphi(\tau)$. On differentiating, we find that

$$f'(\tau) = \left(\int_0^\tau M(\tau) + \varkappa^2 - \tau M(\tau)\right) / \nu^2\varphi^2(\tau).$$

Let τ_0 be the root of equation (1.23). If we analyze the expression for the derivative $f'(\tau)$, it is easy to verify that the function $f(\tau) > 0$ when $\tau > 0$ has a single maximum $\nu_0 = f(\tau_0)$, thus, if $\nu \leq \nu_0$, then the straight line $f = \nu$ intersects on the segment $[0, \tau_0]$ the graph of the function $f = f(\tau)$ at only one point $(\tau_*, f(\tau_*))$, $\tau_* \leqslant \tau_0$ (when $\tau^* = \tau_0$, the straight line $f = \nu$ is tangent to the curve $f = f(\tau)$). Consequently, the equation $\nu = f(\tau)$ has the single root $\tau^* = \tau_0$ on the segment $[0, \tau_0]$. Now, we write equation (1.24) in the form $\tau = \varphi(\tau)$. Since $\varphi'(\tau) = \nu M(\tau) > 0$, then, following the method of L. V. Kantorovich and G. P. Alikov ([6], Ch. XVIII, §1), it is easy to show that the sequence (1.27) converges to τ^*; at the same time, $\tau_n' < \tau^*$. Considering the difference $\tau_* - \tau = \varphi(\tau_*) - \varphi(\tau)$ and making use of the mean value theorem, we have

$$\tau_* - \tau_n' = \varphi'(\tau')(\tau_* - \tau_{n-1}'), \quad \tau_* < \tau' < \tau_{n-1}'$$

$$\text{or} \quad \tau_* - \tau_n' = \nu M(\tau')(\tau_* - \tau_{n-1}').$$

From this, $\tau_* - \tau_n' \leqslant h_0(\tau_* - \tau_{n-1}')$, since $M(\tau)$ is an increasing function and $\nu M(\tau_0) = h_0$. Applying the last bound to $\tau^* - \tau_{n-1}'$, and continuing in this way, we ultimately obtain the bound for (1.28). The lemma is proved.

The following lemma holds for the least root of equation (1.26).

Lemma 2. If

$$\nu \leqslant \nu_0' = \frac{\tau_0'\nu'}{\psi(\tau_0') + \tau_0'} = \left(\int_0^{\tau_0'} N(\tau)\,d\tau\right)^{-1},$$

where τ_0' is a root of equation (1.25) and $\psi(\tau)$ is a function contained in the right-hand side of equation (1.26), then equation (1.26) has on the segment $[0, \tau_0']$ the single root τ_*' to which the following sequence converges:

$$\tau_0 = 0, \qquad \tau_{n+1} = \tau_n + \psi(\tau_n) \qquad (n = 0, 1, \ldots). \tag{1.29}$$

The rate of convergence of the last equation is characterized by the inequality

$$\tau_*' - \tau_n \leqslant (h_0')^n \tau_*',$$

$$h_0' = \lambda\eta_0\eta'\left(\int_0^{\tau_0'} N(\tau)\,d\tau\right)^{-1} \qquad (n = 0, 1, \ldots). \tag{1.30}$$

It is easy to reduce the proof of this lemma to the proof of Lemma 1 if we introduce the functions

$$M^*(\tau) = \int_0^\tau N(\tau)\,d\tau, \quad \varphi^*(\tau) = \psi(\tau) + \tau,$$

by means of which equations (1.16) can be written in the form

$$\tau = \varphi^*(\tau), \qquad \varphi^*(\tau) = \nu'\left(\int_0^\tau M^*(\tau)d\tau + \varkappa'^2\right).$$

1.7. The iteration method. The iteration method is one of the most important methods for solving functional equations. As applied to equation (A), this method consists in constructing the solution of Eq. (A) with the aid of the sequence

$$u_0(\xi) \equiv 0, \quad u_{n+1}(\xi) = S(u_n \mid \xi) \qquad (n = 0, 1, \ldots). \tag{1.31}$$

We can point out two methods for investigating the convergence of the sequence (1.31) to the solution of Eq. (A) and establishing the uniqueness of the obtained solution. The following theorems substantiate the methods.

Theorem 2. Let the given functions included in Eq. (A) satisfy the conditions of Theorem 1 and, in addition, the functions $U(u + \alpha(\xi))$ and $V(u + \beta(\xi))$ are continuously differentiable in respect to u on the segment (1.12) for any ξ on $[-1, 1]$. Further, let $M(\tau)$ be a majorant (1.8) of Eq. (A), and τ° the root of Eq. (1.22). Then, if the quantity $r \geq r^\circ = \sigma\tau^\circ$ included in inequality (1.12) and

$$\lambda \leqslant \frac{1}{\eta}\,\frac{1}{M(\tau^\circ) + \varkappa\sqrt{M'(\tau^\circ)}}, \tag{1.32}$$

then there exists in sphere Ω_{r° the single solution $u^*(\xi)$ of Eq. (A) satisfying condition (1.2). This solution can be obtained as the limit of sequence (1.31). The rate of convergence of this sequence is characterized by the inequality

$$\| u^* - u_n \|_\rho \leqslant (h^\circ)^n\, r^\circ, \quad h^\circ = \lambda\eta M(\tau^\circ),$$

$$(n = 0, 1, \ldots). \tag{1.33}$$

Proof. According to Theorem 1, when all conditions imposed on the given functions included in equation (A) are satisfied, the operator S(u) operates on Ω_{r° in C_ρ^1, and at each point of Ω_{r° there exists a Frechet derivative for which (1.14) holds. Let u(ξ) and v(ξ) be two elements of Ω_{r°. Then, according to the mean value theorem [6]

$$\|S(u)-S(v)\|_\rho \leqslant \|u-v\|_\rho \sup \|S'(v+\theta(u-v))\|_\rho, \quad (0<\theta<1).$$

From this, making use of the bound (1.23) and taking account of the fact that M(τ) is a monotonically increasing function, thus $M(\|u\|_\rho/\sigma) \leqslant M(\tau^\circ)$, if $u(\xi)\in\Omega_{r^\circ}$, and the fact that, on the strength of inequality (1.32), $h^\circ=\lambda\eta M(\tau^\circ)<1$, we find that

$$\|S(u)-S(v)\|_\rho \leqslant h^\circ\|u-v\|_\rho,\ h^\circ<1, \quad \text{if } u(\xi),\ v(\xi)\in\Omega_{r^\circ}. \tag{1.34}$$

Let u(ξ) be an arbitrary element of Ω_{r°. Then, making use of (1.30), (1.27) and the first relationship of (1.11), and taking into consideration that τ° is a root of equation (1.12), we have

$$\|S(u)\|_\rho \leqslant \|S(u)-S(0)\|_\rho + \|S(0)\|_\rho \leqslant h^\circ\|u-v\|_\rho + |\lambda\gamma U(\alpha)V(\beta)+\delta|_\rho \leqslant h^\circ r^\circ + \varkappa^2\lambda\sigma\eta = h^\circ r^\circ + \tau^\circ\sigma\sqrt{M'(\tau^\circ)}\lambda\eta \leqslant h^\circ r^\circ + r^\circ(1-h^\circ),$$

that is,

$$\|S(u)\|_\rho \leqslant r^\circ, \quad \text{if} \quad u(\xi)\in\Omega_{r^\circ}.$$

It follows from inequalities (1.34) and (1.35) that the operator S(u) is the contraction in the sphere Ω_{r° and transforms the sphere Ω_{r° into itself, that is, S(u) satisfies the conditions of the well-known theorem on the principle of linear contractions [5], which then implies Theorem 2.

Theorem 3. Let the given functions included in Eq. (A) satisfy the conditions of Theorem 1 and, in addition, the functions $U(u+\alpha(\xi))$ and $V(u+\beta(\xi))$ and continuously differentiable in respect to u on the segment (1.12) for any ξ on [−1, 1]. Further, let M(τ) be a majorant (1.8) of Eq. (A), and τ_0 a root of Eq. (1.23). Then, if the quantity $r \geq r_0 = \sigma\tau_0$ included in inequality (1.12) and

$$\lambda \leqslant 1/\eta M(\tau_0), \tag{1.36}$$

then the unique solution $u^*(\xi)$ of Eq. (A) satisfying condition (1.2) exists in the sphere Ω_{r_*}, $r_*=\sigma\tau_*$, where τ^* is the least root of Eq. (1.24). This solution can be obtained as the limit of the sequence (1.31), whose rate of convergence is bounded by the inequality

$$\|u^*-u_n\|_\rho \leqslant h_0^n\sigma\tau_*, \quad h_0=\lambda\eta M(\tau_0), \quad (n=0,1,\ldots). \tag{1.37}$$

Proof. According to Theorem 1, there exists in Ω_{r° a continuous arbitrary operator S(u) for which (1.16) holds. We introduce the variable $t=\sigma\tau$ and the function $\varphi_*(t)=\sigma\varphi(t/\sigma)$, where $\varphi(\tau)$ is a function included in (1.24). Since

$$\varphi_*(0)=\sigma\varphi(0)=\sigma\nu\varkappa^2=|\lambda\gamma U(\alpha)V(\beta)+\delta|_\rho,$$

$$\varphi_*'(t)=\varphi'(\tau)=\nu M(\tau),$$

then, making use of inequality (1.16), it is not difficult to establish that

$$\|S(0)\|_\rho=\varphi_*(0), \qquad \|S'(u)\|_\rho \leqslant \varphi_*'(t), \quad \text{if} \quad \|u\|_\rho<t \quad (0\leqslant t<\infty), \tag{1.38}$$

that is, (according to the terminology accepted in functional analysis [6]), the function $\varphi^*(t)$ majorizes the operator S(u). After noting that (1.36) is equivalent to $\nu\leqslant\nu_0=1/M(\tau_0)$, we consider equation $t=\varphi_*(t)$, $0\leqslant t<\infty$. On replacing $t=\sigma\tau$, this equation is transformed into (1.24), thus, according to Lemma 1, it has the single root $r_0=\sigma\tau_0$ on the segment $t_*=\sigma\tau_*$, where τ^* is the limit of the sequence (1.27).

Thus, the operator S(u) has a continuous derivative in the sphere Ω_{r_0}, and the function $\varphi^*(t)$ majorizes the operator S(u) on the segment $[0, r_0]$; at the same time, the equation $t=\varphi^*(t)$ has a single root t^* on the segment $[0, r_0]$. Thus, on the basis of a theorem proved by L. V. Kantorovich ([6], Ch. XVIII, §1), it follows that the equation u = S(u), that is, equation (A) has the solution $u^*(\xi)$ in the sphere Ω_{r^*}; at the same time, the sequence (1.27) converges to this solution. The rate of convergence of (1.27) is bounded by the inequality

$$\|u^*-u_n\|_\rho \leqslant t_*-t_n, \quad t_{n+1}=\varphi_*(t_n) \quad (n=0\ 1,\ldots).$$

Taking into consideration that $t_*=\sigma\tau_*$, $t_n=\sigma\tau_n'$, where τ_n' are members of sequence (1.27), the last inequality can be written in the form

$$\|u^*-u_n\|_\rho \leqslant \sigma(\tau_*-\tau_n') \quad (n=0.\ 1,\ldots).$$

It is easy to obtain the bound (1.37) from this, with the aid of inequality (1.28). It remains for us to prove the uniqueness of the solution. To do this, on the strength of the other theorem proved by L. V. Kantorovich (ibidem), it is sufficient to establish the inequality $\varphi_*(r_0)\leqslant r_0$, which is equivalent to the inequality $\varphi(\tau_0)\leq\tau_0$. However, it is easy to see that the latter is implied by inequality (1.36) since $\varphi(\tau_0)=\tau_0\nu M(\tau')=\tau_0\lambda\eta M(\tau_0)$. The theorem is proved.

Thus, the first method of investigating the convergence of the iteration method and establishing the uniqueness of the obtained solution consists in constructing the majorizing functions of equation (A), determining the parameters η, ϰ, and the majorant M(τ), then computing the root τ° of equation (1.22), and the inequality (1.32) is verified. The second method differs from the first in that the root τ_0 of equation (1.23) is computed and inequality (1.37) is verified. It should be noted that the first method is an extension of the method of A. I. Nekrasov well known in the theory of jets [8, 4] to equation (A) by which a number of problems of detached flows around obstacles with slight curvature have been solved.

It can be seen from the theorems which have been proved that the iteration method can be realized only when the values of the parameters λ, η, ϰ are sufficiently small. However, the parameters η and ϰ which depend on the properties of given functions of equation (A) are actual given quantities in problems of the theory of jets. On the other hand, the parameter λ can turn out to be a quantity which can be given in a whole class of problems and, consequently, λ can always be chosen so small (however, this will have a completely definite physical sense) that the iteration method can be applied to obtain a solution of equation (A). From this viewpoint, the iteration method is the method of the small parameter λ in the theory of jets.

To solve nonlinear functional equations, L. V. Kantorovich developed a method which is known as Newton's method or the method of tangents in the case of ordinary algebraic or transcendental equations [6, 7].

1.8. The Newton-Kantorovich method. As applied to Eq. (A), this method consists in constructing a solution of Eq. (A) with the aid of the sequence

$$u_0(\xi)\equiv 0, \quad u_{n+1}(\xi)=u_n(\xi)-\Gamma(P(u_n)\,|\,\xi), \quad (n=0,1,\ldots), \tag{1.39}$$

in which Γ(P|ξ) and P(u) are operators (1.18) and (1.20). The justification of this method, with some restrictions on the given functions included in Eq. (A), yields the following theorem.

Theorem 4. Let the solution of Eq. (1.18) be represented in the form (1.20) and the conditions of Theorem 1 be satisfied. Moreover, let the functions $U(u + \alpha(\xi))$ and $V(u + \beta(\xi))$ be twice continuously differentiable in respect to u on the segment (1.12) for any ξ on [−1, 1], $N(\tau)$ be the majorant (1.9) of Eq. (A), and τ_0' be a root of Eq. (1.26). Then, if the quantity $r \geq r_0' = \sigma'\tau_0'$ included in inequality (1.12) and

$$\lambda \leqslant 1 / \eta'\eta_0 \int_0^{\tau_0'} N(\tau)\, d\tau \qquad (1.40)$$

in the sphere $\Omega_{r_*'}$, $r_*' = \sigma'\tau_*'$, where τ_*' is the least root of Eq. (1.26), there exists a unique solution of Eq. (A) which satisfies condition (1.2). This solution can be obtained as the limit of sequence (1.39) whose rate of convergence is bounded by the inequality

$$\| u^* - u_n \|_p \leqslant (h_0')^n \sigma'\tau_*', \; h_0' = \lambda\eta'\eta_0 \int_0^{\tau_0'} N(\tau)\, d\tau,$$

$$(n = 0, 1, \ldots) \qquad (1.41)$$

Proof. According to Theorem 1 there exists a second derivative operator $S(u)$ in the sphere $\Omega_{r_0'}$ for which the inequality (1.16) holds. We shall introduce the variable $t = \sigma'\tau$ and the function $\psi_*(t) = \sigma'\psi(t/\sigma')$, where $\psi(\tau)$ is a function included in equation (1.26).

With the aid of bounds (1.21) and (1.16), it is not difficult to show that

$$\psi_*'(0) = -1 < 0, \qquad \| \Gamma(P(0)) \|_p \leqslant \psi_*(0),$$
$$\| \Gamma(P''(u)) \|_p \leqslant \psi_*''(t), \qquad \text{if} \quad \| u \|_p < t, \; 0 \leqslant t < \infty.$$

Here $\| \Gamma(P''(u)) \|_p$ is the norm of the bilinear operator $\Gamma(P''(u)u^*u^{**} \mid \xi)$.

Let us consider equation $\psi_*(t) = 0$, $0 \leqslant t < \infty$. It is easy to see that when $t = \sigma\tau$ is replaced, this equation is transformed to equation (1.26), and inequality (1.40) implies that

$$\nu' \leqslant \nu_0' = \left(\int_0^{\tau_0'} N(\tau) d\tau \right)^{-1},$$

thus, on the basis of Lemma 2, it has a single root $t_* = \sigma\tau_*'$, on the segment $[0, r_0']$, where τ_*' is the limit of the sequence (1.29). Moreover, inequality (1.40) implies that

$$\psi_*(r_0') \leqslant r_0'.$$

If we now turn to the two principal theorems of L. K. Kantorovich on the convergence of Newton's method ([6], Ch. XVIII, §1), it is easy to see that the operator $P(u)$ and the function $\psi^*(t)$ satisfy all conditions of these theorems. It follows from this that equation $P(u) = 0$, that is, equation (A) has a unique solution $u^*(\xi)$ in the sphere $\Omega_{r_*'}$; at the same time, the sequence (1.39) converges to this solution. The rate of convergence of the sequence is bounded by the inequality

$$\| u^* - u_n \|_p \leqslant t_* - t_n, \qquad t_{n+1} = t_n + \psi_*(t_n) \quad (n = 0, 1, \ldots).$$

Taking into consideration that $t_* = \sigma'\tau_*'$, $t_n = \sigma'\tau_n$, where τ_n are the members of sequence (1.29), the last inequality can be written in the form

$$\| u^* - u_n \|_p \leqslant \sigma'(\tau_*' - \tau_n) \quad (n = 0, 1, \ldots).$$

The bound (1.41) is easily obtained from this, with the aid of inequality (1.30). This proves the theorem.

The proved theorem yields a method for investigating the convergence of the Newton-Kantorovich method and establishing the uniqueness of the obtained solution, consisting in that majorizing functions are constructed for Eq. (A), the parameters η', $\varkappa'$, and the majorant $N(\tau)$ are determined, then the root τ_0' of Eq. (1.25) is computed, and inequality (1.40) is verified. Like the iteration method, the Newton-Kantorovich method is the method of the small parameter λ for problems of the theory of jets.

1.9. A linear integro-differential equation. The basic difficulty in practical application of the Newton-Kantorovich method is in solving equation (1.18), that is, in finding the inverse operator $\Gamma(F \mid \xi)$ of the operator $P'(0 \mid \xi)u$. If the expression obtained from formula (1.13) is substituted into equation (1.18) in place of the derivative $S'(0 \mid \xi)u$, then, after differentiating, it will take the form

$$u'(\xi) - \lambda p(\xi)u(\xi) + \lambda q(\xi)T(u \mid \xi) = f(\xi),$$
$$\xi \text{ on } [-1, 1], \qquad (1.42)$$

where

$$p(\xi) = \gamma(\xi)U'(\alpha(\xi))V(\beta(\xi)), \qquad q(\xi) = \gamma(\xi)U(\alpha(\xi))V'(\beta(\xi)),$$
$$f(\xi) = \lambda\gamma(\xi)U(\alpha(\xi))V(\beta(\xi)) + \delta(\xi).$$

It is obvious that the solution of equation (1.19) is equivalent to the solution of equation (1.42) with condition (1.2). It follows from this that if equation (1.42) is solvable for any right-hand side $f(\xi)$ and its solution is represented in the form

$$u = Q(f) = Q(f \mid \xi), \qquad (1.43)$$

where $Q(f \mid \xi)$ is a linear operator, then equation (1.19) is solvable for any right-hand side $F(\xi)$ and its solution is representable through the derivative $F'(\xi)$ in the form

$$u = \Gamma(F \mid \xi) = Q(F' \mid \xi),$$

and the inequality (1.21) is equivalent to the inequality

$$\| Q(f) \|_p \leqslant \eta \, \| F \|_p.$$

Methods for solving equation (1.42) and, consequently, equation (1.19) have not been developed as yet, and this hampers the application of the Newton-Kantorovich method. In special cases, however, when the solution of equation (1.42) can be represented in a simple form, the Newton-Kantorovich method can be applied successfully to solving equation (A). It is possible to propose several methods for reducing equation (1.42) to an equivalent Fredholm equation. We shall present one of them. We shall replace in equation (1.42)

$$v(\xi) = u(\xi)c(\xi), \; s(\xi) = \omega(\xi)c(\xi), \; c(\xi) = \exp\left(-\lambda \int_{\xi_0}^{\xi} p(t) dt\right),$$

$$g(\xi) = f(\xi)c(\xi).$$

Then we obtain the following integro-differential equation for the function $v(\xi)$:

$$v'(\xi) + \lambda q(\xi)T(v, s \mid \xi) = g(\xi), \qquad \xi \text{ on } [-1, 1]$$

the integration of which leads to an equivalent Fredholm equation

$$v(\xi) + \lambda \int_{-1}^{1} K(\xi, t; \lambda)\, v(t)\, dt + g^*(\xi), \qquad \xi \text{ on } [-1, 1],$$

$$K(v, t; \lambda) = \frac{1}{\pi s(t)} \int_{\xi_0}^{\xi} \frac{s(\tau)\, q(\tau)}{\tau - \xi}\, d\tau, \qquad g^*(\xi) = \int_{\xi_0}^{\xi} g(t)\, dt.$$

1.10. The linearization method. A method which makes it possible to find the solution in closed form or in the form of some algorithm that enables one to find the solution with a predetermined accuracy is often called an exact method. From this viewpoint, the iteration method and the Newton-Kantorovich method are exact. The linearization method can be proposed as an approximate method for solving equation (A). It consists in that solving equation (A) is replaced by solving equation (1.42). Then, if the conditions of Theorem 4 are satisfied, the following bound will hold:

$$\|u^* - u^\circ\|_\rho \leqslant \sigma'\tau_*',$$

which follows from inequality (1.41). This bound makes it possible to judge the error which is incurred in solving equation (A) by the linearization method. For an effective solution of equation (A) by the linearization method, it is necessary to have well developed methods for solving equation (1.42).

§2. Jet flows with a curvilinear wall and jet flows of a heavy fluid with rectilinear boundaries. 2.1. Jet flows with a curvilinear wall. Let us consider a steady jet flow of a weightless ideal incompressible fluid whose boundaries consist of a finite number of rectilinear walls, one curvilinear wall, and a free jet converging with the wall, with no stagnation points on the curvilinear wall. It is shown in reference [1] that determination of such a flow is reduced to solving the integro-differential equation

$$u'(\xi) = \lambda\gamma(\xi)K(u(\xi))\exp) - I(u\,|\,\xi)),\quad \xi \text{ on } [-1,1] \qquad (2.1)$$

with the condition u(0) = 0. In the given equation, K(u) is the so-called relative curvature [1] depending only on the form of the curvilinear wall, $\gamma(\xi)$ is a given function depending on the geometrical and physical properties of the flow, and λ is a constant parameter. By the method of construction, the quantity K(u) is an even function, with K(0) = 1. We shall assume further that

$$|K(u)| \leqslant K_0, \qquad |K'(u)| \leqslant K_0', \qquad |K''(u)| \leqslant K_0''. \qquad (2.2)$$

When inequalities (2.2) are satisfied, the functions $U_\nu(t) \equiv K_0{}^\gamma$, $V_\nu(t) = e^t$ ($\nu = 0, 1, 2$) can be taken as majorizing functions of Eq. (2.1). In this case, the basic parameters of Eq. (2.1) will be

$$\eta = |\gamma|_\rho(aK_0' + bK_0), \qquad \varkappa^2 = \frac{b}{aK' + bK_0},$$

$$\eta' = \frac{1}{b}|\gamma|_\rho(a^2K_0'' + 2abK_0' + b^2K_0),$$

$$\varkappa'^2 = \frac{b^2}{a^2K_0'' + 2abK_0' + b^2K_0},$$

and the majorants will be the exponential functions

$$M(\tau) = e^\tau, \qquad N(\tau) = e^\tau, \qquad (2.3)$$

with $\sigma = \sigma' = 1/b$. The majorizing equations (1.12) and (1.16) take the following form after the majorant (2.5) is substituted in them:

$$L(\varkappa, \tau) \equiv \tau - \varkappa e^{-1/2\tau} = 0,$$

$$L_1(\varkappa, \tau) \equiv \tau - 1 + (1 - \varkappa^2)e^{-\tau} = 0,$$

$$L_2(\nu, \varkappa; \tau) \equiv \tau - \nu(e^\tau - 1 + \varkappa^2) = 0,$$

$$\nu = \lambda\eta;\ L_1(\varkappa', \tau) = 0,$$

$$L_3(\nu', \varkappa'; \tau) \equiv \tau - \nu'(e^\tau - 1 - \tau + \varkappa'^2) = 0,$$

$$\nu' = \eta'\eta_0\lambda. \qquad (2.4)$$

Equation (2.1) corresponds to the linear integro-differential equation

$$u'(\xi) - \lambda\gamma(\xi)I(u\,|\,\xi) = f(\xi),$$

$$f(\xi) = \lambda\gamma(\xi), \qquad \xi \text{ on } [-1, 1]. \qquad (2.5)$$

The roots of equations $L(\varkappa, \tau) = 0$, $L_1(\varkappa, \tau) = 0$, $L_1(\varkappa', \tau) = 0$ will be denoted henceforth by τ°, τ_0, τ_0', respectively; and the least roots of equations $L_2(\nu, \varkappa; \tau) = 0$ and $L_3(\nu', \varkappa'; \tau) = 0$ by τ_* and τ_*'.

We now apply Theorems 2–4 to Eq. (2.1). Then it is easy to see that when the value of the parameter λ is sufficiently small, it is possible to obtain a solution of Eq. (2.1) by the iteration method or the Newton-Kantorovich method. Thus, the following assertions hold.

1°. Let the function $\gamma(\xi)$ belongs to class C_ρ and the relative curvature K(u) be continuously differentiable on the segment $t' \leqslant u \leqslant t'$, $t' = mr$, $m = \max(a, b)$. Then, there may be two subcases.

(1) If $r \geq r^\circ = \tau^\circ/b$ and $\lambda \leqslant 1/\eta\,(e^{\tau^\circ} + \varkappa e^{1/2\tau^\circ})$, then there exists the unique solution $u^*(\xi)$ of Eq. (2.1) in the sphere Ω_{r° which satisfies condition u(0) = 0. This solution may be obtained as the limit of the sequence

$$u_0(\xi) \equiv 0,$$

$$u_{n+1}(\xi) = \lambda\int_{\xi_0}^{\xi}\gamma(t)K(u_n(t))\exp(-I(u_n\,|\,t))\,dt,$$

$$(n = 0, 1, \ldots), \qquad (2.6)$$

whose rate of convergence is bounded by the inequality

$$\|u^* - u_n\|_\rho \leqslant (h^\circ)^n r^\circ, \quad h^\circ = \lambda\eta e^{\tau^\circ} \quad (n = 0, 1, \ldots). \qquad (2.7)$$

(2) If $r \geqslant r_0 = \tau_0/b$ and $\lambda \leqslant 1/\eta e^{\tau_0}$, then there exists the unique solution $u^*(\xi)$ of Eq. (2.1) in the sphere Ω_{r^*} which satisfies the condition u(0) = 0. This solution may be obtained as the limit of sequence (2.8) whose rate of convergence is bounded by the inequality

$$\|u^* - u_n\|_\rho \leqslant h_0{}^n r_0, \quad h_0 = \lambda\eta e^{\tau_0} \quad (n = 0, 1, \ldots). \qquad (2.8)$$

2°. Let the solution of Eq. (2.7) be represented in the form (1.43) and let the function $\gamma(\xi)$ belong to class C_ρ, and the relative curvature K(u) be twice continuously differentiable on the segment $t' \leqslant u \leqslant t'$, $t' = mr$, $m = \max(a, b)$. Then, if $r \geqslant r_0' = \tau_0'/b$ and $\lambda < 1/\eta'\eta_0(e^{\tau_0'} - 1)$, there exists the unique solution $u^*(\xi)$ of Eq. (2.1) in the sphere $\Omega_{r'_*}$, $r_*' = \tau_*'/b$ which satisfies the condition u(0) = 0.

This solution may be obtained as the limit of sequence (1.39) in which $\Gamma(F\,|\,\xi) = Q(F'\,|\,\xi)$, where $Q(f\,|\,\xi)$ is a solution of Eq. (2.5), and $P(u\,|\,\xi)$ is an operator of the form

$$P(u\,|\,\xi) \equiv u(\xi) - \lambda\int_{\xi_0}^{\xi}\gamma(t)K(u(t))\exp(-I(u\,|\,t))\,dt.$$

In this case, the rate of convergence of this sequence is bounded by the inequality

$$\|u^* - u_n\|_\rho \leqslant (h_0')^n r_*', \quad h_0' = \lambda\eta'\eta_0(e^{\tau_0'} - 1),$$

$$(n = 0, 1, \ldots). \qquad (2.9)$$

2.2. Jet flows of a heavy fluid with rectilinear boundaries. We shall consider flows of a heavy fluid whose boundaries consist of a finite number of rectilinear solid walls and one free surface. As shown in reference [1], the determination of such a flow is reduced to solving the integro-differential equation

$$u'(\xi) = \lambda\gamma(\xi)\sin(-I(u\,|\,\xi) + \beta(\xi))\exp(-3u(\xi)), \quad \xi \text{ on } [-1, 1] \qquad (2.10)$$

with condition (1.2). In the given equation, λ is a constant parameter, $\gamma(\xi)$ and $\beta(\xi)$ are given functions which depend on the geometric and physical properties of the flow.

It is easy to see that the functions

$$U_\nu(t) = 3\nu e^{3t}, \quad V_\nu(t) \equiv 1 \quad (\nu = 0, 1, 2)$$

can be taken as majorizing functions of Eq. (2.10).

Table 1

$\varkappa$	τ°	τ_0
0.0	0.0	0.0
0.1	0.0953	0.1350
0.2	0.1825	0.2592
0.3	0.2630	0.3738
0.4	0.3378	0.4805
0.5	0.4078	0.5801
0.6	0.4735	0.6737
0.7	0.5355	0.7620
0.8	0.5943	0.8454
0.9	0.6502	0.9246
1.0	0.7035	1.0

In this case, the basic parameters of Eq. (2.10) will be the quantities

$$\eta = |\gamma|_\rho (3a + b), \qquad \varkappa^2 = \frac{3a\,|\gamma \sin\beta|_\rho}{(3a+b)\,|\gamma|_\rho},$$

$$\eta' = \frac{(3a+b)^2\,|\gamma|_\rho}{3a}, \qquad \varkappa'^2 = \frac{9a^2\,|\gamma\sin\beta|_\rho}{(3a+b)^2\,|\gamma|_\rho},$$

and the exponential functions (2.3) will be the majorants; at the same time, $\sigma = \sigma' = 1/3a$. The majorizing equations, consequently, are of the form (2.4).

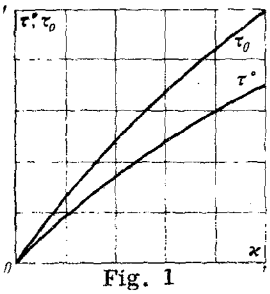

Fig. 1

Equation (2.10) corresponds to the linear integro-differential equation

$$u'(\xi) + \lambda\gamma(\xi)\,[3\sin\beta(\xi)u(\xi) + \cos\beta(\xi)I(u\,|\,\xi)] = f(\xi),$$
$$f(\xi) = \lambda\gamma(\xi)\sin\beta(\xi), \quad \xi \text{ on } [-1, 1]. \qquad (2.11)$$

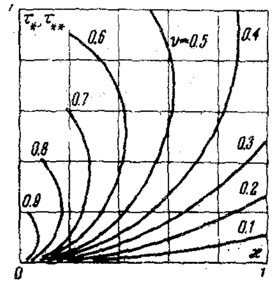

Fig. 2

As before, we shall denote the roots of Eqs. (2.4) by τ°, τ_0, τ_0', τ_*, τ_*'. If we apply Theorems 2–4 to Eq. (2.10), it is easy to see that the iteration method and the Newton–Kantorovich method can be applied to this equation when the value of the parameter λ is small. Thus, the following assertion holds.

1°. Let the function $\gamma(\xi)$ belong to class C_ρ and the function $\beta(\xi)$ be not continuous on the segment $[-1, 1]$. Then we have two subcases:

(1) If $\lambda < 1/\eta\,(e^{\tau^\circ} + \varkappa e^{1/2\tau^\circ})$, then there exists the unique solution $u^*(\xi)$ of Eq. (2.10) in the sphere Ω_{r°, $r^\circ = \tau^\circ/3a$ which satisfies condition (1.2). This solution can be obtained as the limit of the sequence

$$u_0(\xi) \equiv 0, \quad u_{n+1}(\xi) = \lambda\int_{\xi_0}^{\xi}\gamma(t)\sin(-I(u_n\,|\,t) + \beta(t))\exp(-3u_n(t))\,dt, \quad (n = 0, 1, \ldots), \qquad (2.12)$$

whose rate of convergence is bounded by (2.7), where $r_* = \tau_*/3a$.

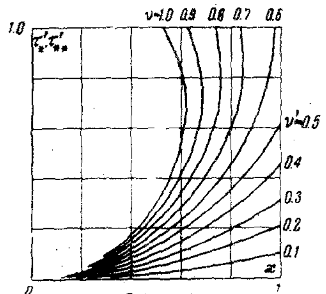

Fig. 3

(2) If $\lambda < 1/\eta e^{\tau_*}$, then there exists the unique solution of Eq. (2.10) in the sphere Ω_{r_*}, $r_* = \tau_*/3a$ which satisfies condition (1.2). This solution can be obtained

Table 2
Values of τ_*

$\varkappa$ \ ν	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.0011	0.0025	0.0043	0.0067	0.0100	0.0152	0.0240	0.0439	
0.2	0.0044	0.0100	0.0172	0.0269	0.0408	0.0630	0.1072		
0.3	0.0100	0.0226	0.0389	0.0613	0.0946	0.1537			
0.4	0.0178	0.0402	0.0696	0.1109	0.1765	0.3335			
0.5	0.0278	0.0630	0.1098	0.1779	0.2998				
0.6	0.0401	0.0911	0.1601	0.2658	0.5265				
0.7	0.0546	0.1245	0.2213	0.3822					
0.8	0.0714	0.1635	0.2949	0.5477					
0.9	0.0905	0.2083	0.3830						
1.0	0.1118	0.2592	0.4894						

Table 3
Values of τ_*'

$\varkappa'$ \ ν'	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.0010	0.0020	0.0030	0.0040	0.0050	0.0060	0.0070	0.0080	0.0090
0.2	0.0040	0.0080	0.0120	1.0160	0.0201	0.0242	0.0283	0.0324	0.0366
0.3	0.0090	0.0180	0.0271	0.0363	0.0455	0.0549	0.0645	0.0743	0.0843
0.4	0.0160	0.0321	0.0483	0.0649	0.0817	0.0990	0.1170	0.1357	0.1555
0.5	0.0250	0.0502	0.0759	0.1022	0.1294	0.1579	0.1882	0.2211	0.4121
0.6	0.0361	0.0725	0.1099	0.1486	0.1896	0.2337	0.2828	0.3399	
0.7	0.0491	0.0990	0.1506	0.2050	0.2641	0.3308	0.4112	0.5236	
0.8	0.0642	0.1298	0.1983	0.2723	0.3557	0.4576	0.6083		
0.9	1.0813	0.1649	0.2535	0.3520	0.4700	0.6398			
1.0	0.1005	0.2045	0.3168	0.4465	0.6191				

as the limit of the sequence (2.12), whose rate of convergence is bounded by (2.8), where $r_* = \tau_* / 3a$.

2°. Let the solution of Eq. (2.10) be represented in the form (1.43) and let the function $\gamma(\xi)$ belong to class C_ρ, and the function $\beta(\xi)$ be continuous on the segment [−1, 1]. Then, if $\lambda < 1 / \eta'\eta_0 (e^{\tau'_0} - 1)$, there exists the unique solution $u^*(\xi)$ of Eq. (2.10) in the sphere $\Omega_{r_*'}$, $r_*' = \tau_*' / 3a$ which satisfies condition (1.2). This solution can be obtained as the limit of sequence (1.39) in which $\Gamma(F \mid \xi) = Q(F' \mid \xi)$, where $Q(f \mid \xi)$ is a solution of Eq. (2.11) and $P(u \mid \xi)$ is an operator of the form

$$P(u \mid \xi) \equiv u(\xi) - \lambda \int_{\xi_0}^{\xi} \gamma(t) \sin(-I(u \mid t) + \beta(t)) \exp(-3u(t))\, dt.$$

In this case, the rate of convergence of this sequence is bounded by inequality (2.9), where $r_*' = \tau_*' / 3a$.

2.3. Computation of the roots of the majorizing equations. The application of the methods set forth here to the two classes of jet flows considered above depends on the satisfaction of inequalities which contain quantities that depend on the roots of the majorizing equations. However, it is easy to see that the majorizing equations are of the very same form (2.4) for flows with a curvilinear wall and for flows of a heavy fluid, it being sufficient to take the four equations

$$L(\varkappa, \tau) = 0, \quad L_1(\varkappa, \tau) = 0,$$
$$L_2(\nu, \varkappa; \tau) = 0, \quad L_3(\nu', \varkappa'; \tau) = 0.$$

Since the equation $L_1(\varkappa', \tau) = 0$ can be derived from the equation $L_1(\varkappa, \tau) = 0$ by replacing $\varkappa$ by $\varkappa'$. Thus, in order to obtain the roots of all majorizing equations, it is sufficient to be able to compute the roots of equations (2.18). In this connection, we present tables of these roots (refer to Table 1 where $\tau°$ is the root of the equation $\tau = \varkappa \exp(-1/_2\, \tau)$, and τ_0 is the root of the equation $\tau = 1 - (1 - \varkappa^2) \exp(-\tau)$; Table 2, where τ^* is the least root of the equation $\tau = \nu (\exp \tau - \tau + \varkappa^2)$; Table 3, where $\tau^{*\prime}$ is the least root of the equation $\tau = \nu'(\exp \tau - 1' - \tau + \varkappa'^2)$, also the graphs of Figs. 1–3, which makes it possible to approximately find the roots of equations (2.13)).

In conclusion, we note that it will be necessary to develop effective exact and approximate methods for calculating the integrals (0.1) and (0.2) to effectively apply these methods to the jet problems considered here. The problem of exact calculation of integrals (0.2) is discussed in reference [9]. In some cases, one can make use of the formulas of reference [10] for approximate computation of the integrals (0.2).

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